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Reproducing Kernel Hilbert Spaces and Optimal State Description of Hadron–Hadron Scattering

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In this paper it is shown that the reproducing kernel Hilbert spaces are adequate variational spaces for the description of the scattering amplitude in terms of a minimum norm principle. Then, the optimal scattering states as the solutions of the minimum norm problems are introduced and the essential characteristic features of the hadron–hadron scattering in the optimal state dominance limit are established.

1. INTRODUCTION

Reproducing kernels have been introduced by Aronszajn (1943, 1950) and Bergman (1950). In fact, a general theory of Hilbert spaces of functions that possess a reproducing kernel has been developed by Aronszajn (1943, 1950). Their usefulness has been demonstrated in the field of conformal mapping (Bergman, 1950; Nehari, 1952), of partial differential equations (Bergman and Schiffer, 1953), numerical analysis (Davis, 1963; Meschkowski, 1962; Golomb and Weinberger, 1958; Shapiro, 1961; Chalmers, 1969; Larkin, 1970; Richter, 1971; Mansfield, 1971), etc. Many additional results of special interest for applications are given by Meschkowski (1962), Shapiro (1971), and Hille (1972). The investigation of the representations of groups which acts in the reproducing kernel Hilbert spaces (RKHS) was stimulated by (see Carey, 1977, 1978) the paper of Perelomov (1972) in which he points out that the notion of "coherent state" (Glauber, 1963a, 1963b; Klauder and McKenna, 1964; Klauder and Sudarshan, 1968; Bargmann, 1961) and the reproducing kernel are the same. On the other hand, Cutkosky (1973) has shown that for applications to the data analysis in the high-energy physics it is important to be able to construct reproducing

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kernels which embody the information which are available from general considerations about the physics of problem. Also, Okubo (1974) proved that the dispersion inequalities for various problems in the elementary particle physics can be treated in unified and generalized manner by using the theory of RKHS.

Recently, by introducing the fluctuations in partial scattering amplitudes, the experimental consequences of the existence of an upper bound L on the scattering angular momenta was investigated (Ion, 1981a). Then, it was shown that there exists a class of phenomena called dual diffractive scattering (DDS) characterized by diffractive pattern very sensitive to the cut-off parameter L. Moreover, assuming that the scattering behaves so as to *maximize the forward unpolarized differential cross section* $(d\sigma/d\Omega)(0^0)$ *when the elastic integrated unpolarized cross section* σ_{el} *is fixed* then it was proved (Ion, 1982) that the DDS states as well as the dual diffractive resonances (Ion, 1981b; Ion and Ion-Mihai, 1981) are zero fluctuating scattering phenomena which saturate the optimal bound

$$\frac{d\sigma}{d\Omega}(0^0) \leq \frac{\sigma_{el}}{4\pi}(L+1)^2 \tag{1}$$

In the present paper the scattering amplitude is considered as an element of a RKHS and then we attempt to indicate the unified manner in which the above results can be described by the reproducing kernels. Since in this paper an extensive use of reproducing kernels will be made, it has been found *necessary* to collect in Section 2 the main properties of the reproducing kernel functions. The general solutions of some important extremals problems are given in Section 3. Concrete applications to the case when the scattering amplitude is an element of a finite-dimensional subspace of $L^2(-1,+1)$ are presented in Section 4 while the conclusions are summarized in Section 5.

2. THE BASIC PROPERTIES OF REPRODUCING KERNELS

Let us consider the two-body elastic scattering process

$$a+b \to a+b \tag{2}$$

where the particles a and b are spinless. Let $x = \cos\theta$ where θ is the c.m. (center-of-mass) scattering angle. Let $f(x)$ be the scattering amplitude of the process (2), with the normalization chosen in such a way that the differential cross section $(d\sigma/d\Omega)(x)$ and the elastic integrated cross section

σ_{el} are given by

$$\frac{d\sigma}{d\Omega}(x) = |f(x)|^2, \qquad x \in [-1, +1] \tag{3}$$

$$\sigma_{\text{el}} = 2\pi \int_{-1}^{+1} \frac{d\sigma}{d\Omega}(x)\, dx = 2\pi \int_{-1}^{+1} |f(x)|^2\, dx = 2\pi \|f\|^2 \tag{4}$$

Since we will work at fixed energy, the dependence of $(d\sigma/d\Omega)(x)$ and σ_{el} on this variable was suppressed.

Therefore, the underlying idea is to consider the scattering amplitude f as an element of a reproducing kernel Hilbert function space H defined on the interval $[-1, +1]$ with the inner product and the norm given by

$$(f, g) = \int_{-1}^{+1} \overline{f(x)} g(x)\, dx, \qquad f, g \in H \tag{5a}$$

$$\|f\|^2 = (f, f) = \int_{-1}^{+1} |f(x)|^2\, dx, \qquad f \in H \tag{5b}$$

Definition 1. A Hilbert space H of complex-valued functions defined on some set S is called reproducing kernel Hilbert space (RKHS for short) if for each fixed point $y \in S$ there exists a unique element $K_y \in H$ such that

$$(f, K_y) = f(y) \qquad \text{all } f \in H \tag{6}$$

K_y is called the *reproducing element* for the point y. The totality of reproducing elements is the *reproducing kernel* (RK) of H or what is equivalent, *RK* of H is the map

$$(x, y) \to K_y(x) = K(x, y) \tag{7}$$

from $S \times S$ to the complex numbers.

Before pointing out some of the properties of RKHS we give a theorem which tells us when a Hilbert function space will have an RK.

Theorem 1. A Hilbert function space H is an RKHS if and only if the evaluation functional is bounded on H, i.e.,

$$|f(y)| \leq C_y \|f\| \tag{8}$$

for all $f \in H$, where $C_y \geq 0$ depend only on y.

Proof. If H is an RKHS, then

$$|f(y)| = |(f, K_y)| \leq \|f\| \cdot \|K_y\| \tag{9}$$

so the evaluation functional is bounded with $C_y = \|K_y\| = [K(y, y)]^{1/2}$.

Let the evaluation functional be bounded. Then by Frechet–Riesz representation theorem [see Davis (1963), Theorem 9.3.3] for $y \in S$ there exist a unique element $g_y \in H$ such that $f(x) = (f, g_y)$. Therefore g_y has required reproducing property.

The RKHS has the following useful properties.

(a) The reproducing kernel, if it exists, is unique.

(b) $K(x, y) = \overline{K(y, x)}$ (Hermitian symmetry) (10)

(c) $|K(x, y)|^2 \leq K(x, x)K(y, y)$; $\|K_y\|^2 = K(y, y) \geq 0$ (11)

(d) If H_0 is a subspace of the reproducing kernel Hilbert space H then H_0 is also RKHS with the kernel defined by

$$K_y(x) = (PK_y)(x) \tag{12}$$

where P is the orthogonal projection from H onto H_0.

(e) If the reproducing kernel Hilbert space H is a subspace of a larger space H_1 then the projection from H_1 onto H is given by

$$(Pf)(y) = (K_y, f), \qquad f \in H_1 \tag{13}$$

(f) If H is a RKHS with reproducing kernel K, then for all $f \in H$ and $y \in S$

$$|f(y)| \leq [K(y, y)]^{1/2} \|f\| \tag{14}$$

equality holding if and only if

$$f(x) = f(y) \frac{K(x, y)}{K(y, y)}, \qquad K(y, y) \neq 0 \tag{15}$$

(g) The kernel is a positive definite function, i.e.,

$$\sum_{i,j} \bar{a}_i a_j K(x_i, x_j) \geq 0 \tag{16}$$

for all finite sets $\{a_i\} \subset C$ and $\{x_i\} \subset S$.

(h) If $\{\phi_n\}$ is a complete orthonormal sequence in RKHS, then

$$K(x, y) = \sum_n \bar{\phi}_n(x) \phi_n(y) \tag{17}$$

Corollary 1. If $K(x, y)$ for x and y in $[-1, +1]$ is the reproducing kernel of the Hilbert space H and if the scattering amplitude f is an element of H, then the functionals (3) and (4) must obey the inequality

$$\frac{d\sigma}{d\Omega}(y) \leq \frac{\sigma_{\text{el}}}{2\pi} K(y, y) \tag{18}$$

If $K(y, y) \neq 0$ and $f(y) \neq 0$, then the equality holds in (18) if and only if

$$f(x) = f(y) \frac{K(x, y)}{K(y, y)} = f(y) \frac{\sigma_{el}}{2\pi(d\sigma/d\Omega)(y)} K(x, y) \tag{19}$$

3. THE RKHS AND EXTREMAL PROBLEMS

Now we shall see that the reproducing kernel $K(x, y)$ of the scattering amplitude Hilbert space H can be characterized as the solution to certain extremal problems. Therefore, let us define the following extremal problems.

Problem A. Minimize $\|f\|$ subject to $f \in H$, $f(y) = a$ for $a \in C$ and $y \in [-1, +1]$.

Problem B. Maximize $|f(y)|$ subject to $f \in H$, $y \in [-1, +1]$ and $\|f\| = [\sigma_{el}/2\pi]^{1/2}$ fixed.

Theorem 2. If $K(y, y) \neq 0$, then Problem A has the unique solution $f = bK_y$ with $b = a[K(y, y)]^{-1}$.

Proof. We write f in the form

$$f = bK_y + h, \qquad h \in H$$

Then, the condition $f(y) = a$ gives

$$a = bK(y, y) + h(y)$$

and hence $h(y) = 0$. Moreover

$$\|f\|^2 = (bK_y, bK_y) + \bar{b}(K_y, h) + b(h, K_y) + (h, h)$$

Since $(K_y, h) = h(y) = 0$, we have

$$\|f\|^2 = \|bK_y\|^2 + \|h\|^2$$

Therefore $\|f\|$ is minimized when $\|h\| = 0$ and hence $h = 0$. Then $f = bK_y$ with $b = a[K(y, y)]^{-1}$. ■

Theorem 3. If $K(y, y) \neq 0$, then Problem B has a solution uniquely determined in the sense that $f \in H$ is a solution if and only if it has the form $f = bK_y$ with $|b|^2 = \|f\|^2[K(y, y)]^{-1} = (\sigma_{el}/2\pi)[K(y, y)]^{-1}$.

Proof. The result is obtained using Eqs. (14), (15), (3), and (4). ■

The following corollary lists different equivalent extremal properties of the function (19).

Corollary 2. Assuming $K(y, y) \neq 0$, $y \in [-1, +1]$, then (i) functions of form bK_y *minimize* $\|f\| \, |f(y)|^{-1}$ subject to $f \in H$, $f(y) \neq 0$, and *maximize*

$|f(y)|\,\|f\|^{-1}$ subject to $f\in H$, $f\neq 0$, (ii) functions of form bK_y, where $|b|^2=[K(y,y)]^{-1}$, maximize $|f(y)|$ subject to $f\in H$, $\|f\|=1$.

Definition 2. The scattering state described by the amplitude (19) is called *optimal state* for the process (2).

The following extremal problem is a direct generalization of the Problem A.

Problem C. Let $y_1, y_2, \ldots, y_n$, be n distinct points in the interval $[-1,+1]$ and let $a_1, a_2, \ldots, a_n$, be n complex numbers distinct or not. Find the element of **H** of least norm for which $f(y_i)=a_i$, $i=1,2,\ldots,n$.

Problem D. Let $y_1, y_2, \ldots, y_n$ be the points given in Problem C. For each $j=1,2,\ldots,n$, find the element of H of minimum norm for which $f(y_j)=1$, $f(y_k)=0$, $k\neq j$, $k=1,2,\ldots,n$.

Theorem 4. Let Δ_n be the determinant $\Delta_n=\det[K(y_i,y_j)]$, i, $j=1,2,\ldots,n$. For $\Delta_n\neq 0$ the Problem C has the unique solution

$$f(x)=-\Delta_n^{-1}\cdot\begin{vmatrix} 0 & K(x,y_1) & K(x,y_2) & \ldots & K(x,y_n)\\ a_1 & K(y_1,y_1) & K(y_1,y_2) & \ldots & K(y_1,y_n)\\ a_2 & K(y_2,y_1) & K(y_2,y_2) & \ldots & K(y_2,y_n)\\ \vdots & \vdots & \vdots & & \vdots\\ a_n & K(y_n,y_1) & K(y_n,y_2) & \ldots & K(y_n,y_n)\end{vmatrix}$$

$$=\sum_{j=1}^{n} a_j\frac{\Delta_{nj}(x)}{\Delta_n} \tag{20}$$

and

$$\|f\|^2=-\Delta_n^{-1}\cdot\begin{vmatrix} 0 & \bar{a}_1 & \bar{a}_2 & \ldots & \bar{a}_n\\ a_1 & K(y_1,y_1) & K(y_1,y_2) & \ldots & K(y_1,y_n)\\ a_2 & K(y_2,y_1) & K(y_2,y_2) & \ldots & K(y_2,y_n)\\ \vdots & \vdots & \vdots & & \vdots\\ a_n & K(y_n,y_1) & K(y_n,y_2) & \ldots & K(y_n,y_n)\end{vmatrix} \tag{21}$$

where $\Delta_{nj}(x)$ is the determinant obtained from Δ_n by the substitution of the row j with the elements $K(x,y_1)$, $K(x,y_2),\ldots,K(x,y_n)$, respectively.

Proof. The subset of H where the elements satisfy the conditions $f(y_i)=a_i$, $i=1,2,\ldots,n$, is evidently closed nonvoid and convex. Then it is easy to prove [see Davis (1963), Lemma 2.13, p. 332] the existence of a unique element of minimum norm. To determine this element and its norm we consider all elements of H of the form

$$g(x)=\sum_{j=1}^{n} b_j K(x, y_j) \tag{22}$$

which form a linear finite-dimensional subspace H_0 of H. Hence for any $f \in H$ we have a unique decomposition

$$f=g+h, \qquad g \in H_0, \qquad h \in H_0^{\perp}$$

where $H_0^{\perp}$ is orthogonal complement of H_0 in H. Since $g \in H_0$ and $h \in H_0^{\perp}$ then

$$(h, K_{yj})=h(y_j)=0, \qquad j=1,2,\ldots,n$$

and

$$f(y_j)=g(y_j)=a_j, \qquad j=1,2,\ldots,n$$

Further, from $(g, h)=0$ we have

$$\|f\|^2=\|g\|^2+\|h\|^2$$

Hence it follows that the minimum is obtained for $\|h\|=0$ and thus $h=0$. Therefore, we have obtained that the function of minimum norm is of form (22) for some particular choice of the complex numbers b_j, $j=1,2,\ldots,n$. Then, these numbers are uniquely determined using the following n linear equations:

$$\sum_{j=1}^{n} b_j K(y_i, y_j)=a_i, \qquad i=1,2,\ldots,n$$

We note that the value of the determinant $\Delta_n=\det[K(y_i, y_j)]$ is real since $K(y_i, y_j)=\overline{K(y_j, y_i)}$. Also, from the property (g) of RK it follows that $\Delta_n \geq 0$. The value $\Delta_n=0$ is excluded since in this case the extremal Problem C has no solution. Once the b_js have been found by the Cramer's rule, then it is easy to find the results (20) and (21). ■

Corollary 3. If $\Delta_n \neq 0$, then (i) Problem D has the unique solution

$$f_j(x)=\frac{\Delta_{nj}(x)}{\Delta_n} \qquad \text{for each } j=1,2,\ldots,n \tag{23}$$

(ii) If the points $y_i \in [-1,+1]$ satisfy the conditions

$$K(y_j, y_j) \neq 0;\ K(y_i, y_j)=K(y_j, y_j)\delta_{ij},\ i,j=1,2,\ldots,n \tag{24}$$

then

$$f_j(x)=\frac{K(x, y_j)}{K(y_j, y_j)}, \qquad j=1,2,\ldots,n \tag{25}$$

and the results (20) and (21) are given by

$$f(x)=\sum_{j=1}^{n} f(y_j)\frac{K(x, y_j)}{K(y_j, y_j)} \tag{26}$$

$$\|f\|^2=\sum_{j=1}^{n} |f(y_j)|^2[K(y_j, y_j)]^{-1} \tag{27}$$

(iii) If $n=\dim H$, then the functions (25) form a complete orthogonal system.

Definition 3. The biorthogonal functions $f_j(x)$ defined by equations (23) are called fundamental optimal states (FOS) of the scattering process (2).

Definition 3 is inspired by the fact that the system of functions (23) represents a generalization of the system of fundamental polynomials used in the Lagrange interpolation [see Davis (1963), p. 33].

The importance of FOS systems lie in the identity

$$f_j(y_k)=\delta_{jk}=\begin{cases}0 & \text{if } j\neq k\\ 1 & \text{if } j=k\end{cases} \tag{28}$$

and the resulting simple explicit solutions (20) [or (26)] and (23) [or (25)] of the extremal Problem C and Problem D, respectively.

4. APPLICATIONS

Let $f(x)$ be the scattering amplitude of the process (2) written in terms of partial amplitude as

$$f(x)=\sum_{l=0}^{L} (2l+1)f_l P_l(x), \qquad x\in[-1,+1], f_l\in C \tag{29}$$

where $P_l(x)$, $l=0, 1, \ldots$ are the Legendre polynomials. Then it is easy to verify that (i) *the scattering amplitude* $f(x)$ *is an element of an RKHS* H *defined on* $[-1,+1]$ *if and only if*

$$L<\infty \tag{30}$$

and that (ii) H is an $(L+1)$-dimensional subspace of $L^2[-1,+1]$ and possesses the reproducing kernel

$$\begin{aligned}K(x, y)&=\sum_{l=0}^{L}\left(l+\frac{1}{2}\right)P_l(x)P_l(y)\\ &=\frac{1}{2}(L+1)\frac{P_{L+1}(x)P_L(y)-P_L(x)P_{L+1}(y)}{x-y}\end{aligned} \tag{31a}$$

$$K(y, y)=\tfrac{1}{2}(L+1)[\dot{P}_{L+1}(y)P_L(y)-\dot{P}_L(y)P_{L+1}(y)] \tag{31b}$$

where $\dot{P}_l(y)=dP_l(x)/dx$.

Corollary 4. Assume that the scattering amplitude $f(x)$ is an element of the Hilbert space H which has the reproducing kernel (31a,b).

(i) Then, if σ_{el} and $(d\sigma/d\Omega)(1)$ are given, any cut-off on the angular momentum must obey the bound

$$(L+1)^2 \geq \frac{4\pi}{\sigma_{el}}\cdot\frac{d\sigma}{d\Omega}(1) \tag{32}$$

(ii) The equality holds in (32) if and only if $f(x)$ is the optimal amplitude

$$f(x)=f(1)\frac{K(x,1)}{K(1,1)}=f(1)\frac{\dot{P}_{L+1}(x)+\dot{P}_L(x)}{(L+1)^2} \tag{33a}$$

where L is

$$L=\text{integer}\left\{\left[\frac{4\pi}{\sigma_{el}}\frac{d\sigma}{d\Omega}(1)\right]^{1/2}-1\right\} \tag{33b}$$

(iii) The logarithmic slope of the forward diffraction peak is given by

$$b_o=\frac{d}{dt}\left[\ln\frac{d\sigma}{d\Omega}(s,t)\right]\bigg|_{t=0}=\frac{\lambda^2}{4}\left[\frac{4\pi}{\sigma_{el}}\cdot\frac{d\sigma}{d\Omega}(1)-1\right] \tag{34}$$

where $\sqrt{s}$ and $\sqrt{|t|}$ are the c.m. energy and transfer momentum, respectively and λ is the c.m. d'Broglie wave length.

(iv) The forward diffraction peak of the optimal phenomena described by equations (33a,b) possesses the scaling property

$$\frac{1}{(d\sigma/d\Omega)(1)}\cdot\frac{d\sigma}{d\Omega}(x)=\left[\frac{J_1(\tau)}{\tau}\right]^2 \quad \text{for } L\gg 1, \text{small } \tau \tag{35a}$$

where

$$\tau=2[|t|b_0]=\left\{\lambda^2|t|\left[\frac{4\pi}{\sigma_{el}}\cdot\frac{d\sigma}{d\Omega}(1)-1\right]\right\}^{1/2} \tag{35b}$$

and $J_1(\tau)$ is the Bessel function of first order.

Also, using the kernel (31a,b), the solution (20), (21) of the extremal Problem C can be written in an explicit form. Here we present this solution only in the particular case when the points $y_j\in[-1,+1]$, $j=1,2,\ldots,L+1$, are the zeros of $P_{L+1}(x)$.

Corollary 5. Assume that the scattering amplitude $f(x)$ is an element of the RKHS with the reproducing kernel (31a,b) and that the points $y_j \in [-1, +1]$ are the zeros of $P_{L+1}(x)$.

(i) Then, the fundamental optimal states (25) are given by

$$\frac{K(x, y_j)}{K(y_j, y_j)} = \frac{P_{L+1}(x)}{(x-y_j)\dot{P}_{L+1}(y_j)}, \qquad j = 1, 2, \ldots, L+1 \tag{36}$$

(ii) The solution of the extremal Problem C is

$$f(x) = \sum_{j=1}^{L+1} f(y_j) \frac{P_{L+1}(x)}{(x-y_j)\dot{P}_{L+1}(y_j)} \tag{37a}$$

where

$$\frac{\sigma_{\text{el}}}{4\pi} = \frac{1}{L+1} \sum_{j=1}^{L+1} \frac{d\sigma}{d\Omega}(y_j) \frac{1}{\dot{P}_{L+1}(y_j) P_L(y_j)} \tag{37b}$$

(iii) The partial amplitudes are expressed in terms of $f(y_j)$, $j = 1, 2, \ldots, L+1$, as follows:

$$f_l = \frac{1}{L+1} \sum_{j=1}^{L+1} f(y_j) \frac{P_l(y_j)}{\dot{P}_{L+1}(y_j) P_L(y_j)}, \qquad l \leq L \tag{38}$$

and $f_l = 0$ for $l > L$.

Remark 1. For the Hilbert space H with the reproducing kernel (31a,b) and for $y_j \in [-1, +1]$ satisfying the equations $P_{L+1}(y_j) = 0, j = 1, 2, \ldots, L+1$, the FOS system is the system of fundamental polynomials [see Davis (1963), p. 33] for the Lagrange interpolation problem when the scattering amplitude $f(x)$ is given in the zeros of the Legendre polynomials.

Remark 2. The predictions given by Corollary 4 are a particular case (the zero-spin scattering case) of the results given in Ion (1982a). All the results of Sections 2 and 3 can be extended to the scattering of particles with arbitrary spins.

5. CONCLUSIONS

Information on the scattering amplitudes can be obtained by assumption that the scattering system (20) behaves as to optimize some given measure of the scattering effectiveness. Then, the behavior of the system can be completely specified by identifying the criterion of effectiveness and applying optimization to it. This approach, known as describing the scattering system in terms of an optimum principle, was illustrated here by using the Hilbert space methods.

The conclusions of this paper may be summarized as follows:

(1^0) The RKHS is an adequate variational space for the description of the scattering amplitude in terms of a *minimum norm principle.*

(2^0) The *optimal scattering states,* introduced here as the solutions of the minimum norm problems are responsible for the diffractive patterns as well as for the dual diffractive behavior (Ion, 1981a; Ion, 1982a,b; Ion, 1981b; Ion and Ion-Mihai, 1981a,b) of the two-body elastic scattering.

(3^0) The expansion (37a) of the scattering amplitude in terms of the FOS system (36) seems to be an important alternative to the partial wave analysis.

(4^0) The optimal state dominance in the hadron–hadron scattering is a fact well established (Ion, 1982) for all pp, $\bar{p}p$, $\pi^{\pm}p$, $k^{\pm}p$ scattering at all energies higher than 2 GeV where the predictions (34) and (35a,b) are satisfied experimentally to a surprising accuracy. Detailed comparisons of the optimal state dominance predictions with the experimental data at low energies are of great theoretical interest since they might provide experimental evidence for a specific interpretation of Morrison D resonances (Morrison, 1970) as optimal resonances (Ion, 1982b).

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